



1. Find, with proof, all integers n such that $2^6 + 2^9 + 2^n$ is the square of an integer.
2. Find, with proof, all real numbers a such that $|x - 1| - |x - 2| + |x - 4| = a$ has exactly 3 solutions.
3. As Lisa hurried to copy down the last problem of her math homework assignment at the end of class, she got as far as

$$0 = 9x^8 - 28x^6 -$$



SOLUTIONS

1. If $m^2 = 2^6 + 2^9 + 2^n = 576 + 2^n = 24^2 + 2^n$, then $2^n = m^2 - 24^2 = (m - 24)(m + 24)$. Therefore, $m + 24$ and $m - 24$ must each be powers of 2. Let $m + 24 = 2^k$ and $m - 24 = 2^p$ where, $p < k$ and $p + k = n$. Then $2^k - 2^p = 48$ which implies 2^p divides 48, so that $p \leq 4$. Trying $p = 0, 1, 2, 3, 4$ gives $p = 4, k = 6$ and $n = 10$ as the only possible value for n .

2. We begin by graphing the function $y = f(x) = |x - 1| - |x - 2| + |x - 4|$. If $x \leq 1$, then $x - 1 \leq 0, x - 2 \leq 0$ and $x - 4 \leq 0$, so we have

$$y = -(x - 1) + (x - 2) - (x - 4) = -x + 3.$$

If $1 \leq x \leq 2$, then $x - 1 \geq 0, x - 2 \leq 0$ and $x - 4 \leq 0$, so

$$y = (x - 1) + (x - 2) - (x - 4) = x + 1.$$

In the interval $2 \leq x \leq 4$ we have $x - 1 \geq 0, x - 2 \geq 0$ but $x - 4 \leq 0$, so

$$y = (x - 1) - (x - 2) - (x - 4) = -x + 5.$$

Finally, for $x \geq 4$ we have

$$y = (x - 1) - (x - 2) + (x - 4) = x - 3.$$

By piecing together the relevant parts of these four linear functions, we get the graph of the function $f(x)$ shown at the right.

Thus, for the original equation to have exactly three solutions, we have to choose a so that the horizontal line $y = a$ touches the graph of $f(x)$ at exactly three points. From the graph, this happens only for $a = 2$ and $a = 3$.

3. $P(x) = 9x^8 - 28x^6 - \dots$ Since the coefficient of x^7 is zero, the sum of the roots of $P(x) = 0$ is zero. Let the seven identical roots be a and the desired eighth root be b . Then $7a + b = 0$ or $b = -7a$. Standardizing $P(x)$, the sum of the products of the roots taken two at a time is $-\frac{28}{9}$. Therefore, $C^2 = 7 \cdot \frac{28}{9}$

4. a. Let the desired pair-square set be $\{2012, a, b\}$. Since $2012 + 13 = 2025 = 45^2$, try $a = 13$ as a second member of the set. Then

$$2012 + b = x^2 \quad \text{and} \quad 13 + b = y^2 \quad \text{for some integers } x \text{ and } y.$$

Subtracting these two equations gives $1999 = x^2 - y^2 = (x + y)(x - y)$. Since 1999 is a prime number, $x + y = 1999$ and $x - y = 1$. From these two equations we obtain $x = 1000$. Then $b = 1000^2 - 2012 = 1000000 - 2012 = 997988$. Thus, one desired pair-square set of size 3 is $\{13, 2012, 997988\}$.

(Note: $\{292, 2012, 45077\}$ and $\{488, 2012, 143912\}$ also work. There are others.)

- b. Every integer has one of the four forms $4k$; $4k + 1$; $4k + 2$ and $4k + 3$ for integers k .

First we prove that the square of an integer must have one of the forms $4n$ or $4n + 1$.

Proof:

$$(i) \quad (4k)^2 = 4(4k^2) = 4n$$

$$(ii) \quad (4k + 1)^2 = 16k^2 + 8k + 1 = 4(4k^2 + 2k) + 1 = 4n + 1$$

$$(iii) \quad (4k + 2)^2 = 16k^2 + 16k + 4 = 4(4k^2 + 4k + 1) = 4n$$

$$(iv) \quad (4k + 3)^2 = 16k^2 + 24k + 9 = 4(4k^2 + 6k + 2) + 1 = 4n + 1$$

Therefore, the square of an integer must have one of the forms $4n$ or $4n + 1$.

Let S be a pair-square set of size 3. Suppose that the set S contains the two odd numbers a and b . Since $a + b$ is an even square, it must have form $4n$, and therefore a and b cannot both have form $4k + 1$, nor can they both have form $4k + 3$. It follows that we can write $a = 4k + 1$ and $b = 4k + 3$.

We derive a contradiction by showing that there is no possibility for the third member z of S . Indeed, if z has form $4k$ or $4k + 3$, then $z + b$ is not a square, and if z has form $4k + 2$ then $z + a$ is not a square.

5. It is easy to show that $\triangle ADP$ is isosceles (note the marked congruent angles).

Thus $DP = 2012$ and all we need is the length of PC .

Since $\triangle ADP$ and $\triangle CBQ$ are congruent isosceles triangles,

$\angle PAB \cong \angle DAP \cong \angle CQB$, making \overline{AP} parallel to \overline{QC} .

Similarly, the other two angle bisectors are parallel, making $EFGH$ a parallelogram.

Since $\angle DAB$ and $\angle ADC$ are supplementary, $m\angle DAB + m\angle ADC = 180$.

Since m