THE 2013–

6. Of the 50 states, there are exactly twice as many larger than Hawaii as there are larger than Florida. There are exactly four times as many states smaller than Florida as there are states smaller than Hawaii. How many states are both larger than Hawaii and smaller than Florida?

(A) 20 (B) 21 (C) 22 (D) 23 (E) 24

7. Two 8 x 12 rectangles share a common corner and overlap as in the diagram, so that the distance, AB, from the bottom right corner of one rectangle to the intersection point B along the right edge of that rectangle is 7. What is the area of the region common to the two rectangles?

(A) 36 (B) 38 (C) 40 (D) 42 (E) 44

8. If the three digit number $\underline{A}\ \underline{B}\ \underline{C}$ is decreased by the sum of its digits, the result is a perfect square. Which of the following is <u>not</u> a possible value for A?

(A) 3 (B) 4 (C) 5 (D) 6 (E) 7

9. Which of the following could not be the discriminant of the quadratic equation

19. How many ordered pairs (x, y) of integers are solutions to the equation $\frac{}{+} =$?

(A) 26 (B) 27 (C) 42 (D) 53 (E) 54

20. If $T\,E\,E\,N$ is a four-digit number such that $T\,E\,E\,N_5 + T\,E\,E\,N_7 = T\,E\,E\,N_8$, what is the value of the digit N?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

21. The vertices of a triangle are (12, 20), (26, 96) and (0, k). What value of k

7. D Constructing line segment $\overline{CB}$

12. C Positive integral powers of 3 end in 3, 9, 7, and 1 only, with equal frequency as the exponent goes from 1 to 100. Consider the table of the sums of the units digits shown. To be a multiple of 5, a zero units digit must be 0 or 5. This occurs 4 times out of 16, for a probability of .

13. C Suppose before scoring 98 and 70, the student's average grade on her first N tests was M. Then $\frac{+}{+} = \quad +$ and $\frac{8 \quad 70}{2} \quad 1 ! 2.$
The first equation simplifies to $M + N = 97$ and the second to $2M - N = 170$. Thus $M = 89$ and $N = 8$. Including the two tests, she took **10** tests.

14. C Any regular polygon can be inscribed in a circle. Consider the the circle at the right in which two adjacent sides of a regular polygon and the angle trisectors of the angle between them have been drawn. Since the three inscribed angles intercept congruent arcs, each of the three arcs must contain the same number of sides of the regular polygon. Let this number of sides be k. Therefore, the number of sides of the regular polygon must be of the form $3k$

17. B Rewrite the formula as $f(n) - f(n-1) = n$. Look at the following pattern:

$f(2013) - f(2012) = 2013$
$f(2012) - f(2011) = 2012$
$f(2011) - f(2010) = 2011$
$f(2010) - f(2009) = 2010$
$f(2009) - f(2008) = 2009$

$f(2) - f(1) = 2$

18. B Let a represent the length of the sides of the equilateral triangles.

Then the lengths of altitudes CE and DE are $\frac{a\sqrt{\ }}{\ }$.

Since CE and DE are in perpendicular planes, triangle CED

is an isosceles right triangle. Therefore, CD = $\frac{a\sqrt{3}}{2}\sqrt{2} = \frac{a\sqrt{6}}{2}$.

Using the Law of Cosines on isosceles triangle ACD,

$\left(\frac{\sqrt{6}}{2}\right)^2 \quad {}^2 \quad {}^2 - 2 \ {}^2 \cos\,!$ from which $\cos$ CAD = $\frac{1}{4}$.

19. D The given equation $\frac{\ }{+} =$ can be rewritten as $xy - 2013x - 2013y = 0$.

Adding 2013^2 to both sides allows us to factor the left side: $(x - 2013)(y - 2013) = 2013^2$.

Letting $(x - 2013) = k$, then $y - 2013 = \frac{2013^2}{k}$. Each solution (x, y) to the original equation yields a different value of k. Since $2013 = (3^1)(11^1)(61^1)$, we know that 2013^2 has $(2 + 1)(2+ 1)(2 + 1) = 27$ positive integral factors. Therefore, there are $2(27) = 54$ integral values of k. However, if $k = -2013$, $x = y = 0$ which does not work in the original equation. Therefore, there are 53 solutions.

20. C Noting that all three digits must be either 0, 1, 2, 3, or 4,

$TEEN_5 = 125T + 25E + 5E + N = 125T + 30E + N$

$TEEN_7 = 343T + 49E + 7E + N = 343T + 56E + N$

$TEEN_8 = 512T + 64E + 8E + N = 512T + 72E + N$

Therefore, $125T + 30E + N + 343T + 56E + N = 512T + 72E + N$, from which

$44T - 14E = N$. If $T = 1$, then $E = 3$ and $N = 2$. If $T = 2$, 3 or 4, then N must be greater than 4 for E to be less than 5. Therefore, N = 2.

21. C Let point A have coordinates (12, 20) and B have coordinates (26, 96). The length of AB is fixed for any choice of *k* on the y-axis. Reflect A over the y-axis to A′(-12, 20) and draw A′B intersecting the y-axis at C. Since AC = A′C, the distance A′B = BC + AC. Since the shortest distance between two points is along a straight line, BC + AC is a minimum. To find *k*, find the equation of A′B (y = 2x + 44) and *k* = 44.

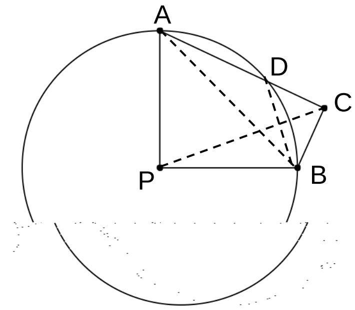
22. A The left expression can be rewritten as $(\frac{3}{2})(\frac{8}{3})(\frac{15}{4})(\frac{24}{5})(\frac{35}{6})!!!$. Dividing each numerator by the denominator of the *preceding* fraction, this expression becomes

– – – – – ... — . If the numerator was multiplied by 2, it would become 61!.

Fortunately, if the denominator is multiplied by 2, it becomes 120 = 5!. Therefore, the original expression is equivalent to —. Hence (a, b) = (61, 5).

23. E To find solutions, note that for $0 < A < 90$, $\cos(A) = \sin(90–A)$. Therefore, solutions may be obtained from $x^2 + x = 90$. This equation has one positive solution, $x = 9$ (the other solution is -10). Also, since $\sin[180– (90–A)] = \sin(90–A) = \cos(A)$, solutions may be obtained from $180 – x^2 + x = 90$ or $x^2 – x - 90 = 0$. This also yields one positive solution, $x = 10$ (the other solution is -9). Hence two solutions are $x = 9, 10$.
To find any other values we need to find all possible integer solutions between 0 and 90 for the equations $x^2 \pm x = 90 + 360K$ (K a positive integer). Since $90 + 360K$ is divisible by 9 and 5 and $x^2 \pm x$ factors as $x(x \pm 1)$, we need to find factor pairs such that x is a multiple of 9 and x, $x + 1$, or $x – 1$ is a multiple of 5. The possibilities (other than (9,10)) are (35,36), (44,45), (45,46) (54,55), and (80,81). Of these pairs, only 54 and 55 have a product of the form $90 + 360K$. Therefore, the only other two solutions are $x = 54, 55$. Thus the sum of all four solutions is 9 + 10 + 54 + 55 = **128**.

25. E Method 1



Construct chord $\overline{\quad}$. Since the measure of minor arc ADB = 90, the measure of major arc AB = 270, making the measure of inscribed angle ADB = 135. Therefore, m CDB = 45, so that Δ DCB is an isosceles right triangle, and DC = CB = 5.

Since both pairs of opposite angles of quadrilateral APBC are supplementary, it is a cyclic quadrilateral.

Although it is possible to compute the radius of circle P ($\sqrt{73}$), it is not necessary. Letting AP = PB = R, and noting that AB = R$\sqrt{2}$, apply Ptolemy’s Theorem.
5R + 11R = PC(R$\sqrt{\ }$). Therefore, PC = $\frac{16}{\sqrt{2}} \quad 8\sqrt{2}$.

Method 2

Construct chord $\overline{\quad}$. Since the measure of minor arc ADB = 90, the measure of major arc AB = 270, making the measure of inscribed angle ADB = 135. Therefore, m CDB = 45, so that Δ DCB is an isosceles right triangle, and DC = CB = 5.

Using the Pythagorean Theorem on ABC, AB = $\sqrt{\quad}$. Since APB is an isosceles right triangle, AP = PB = $\sqrt{73}$.

Since PAC and PBC are supplementary let m PAC = !, and m PBC = 180 – !.
Using the Law of Cosines on both APC and BPC:

(1) $PC^2 = 73 + 121 \quad (2)(11)\sqrt{73}\cos \quad = 194 \quad 22\sqrt{73}\cos$

(2) $PC^2 \quad 73 \quad 25 \quad (2)(5)\sqrt{73}\cos(180 \quad) \quad 98 - 10\sqrt{73}\cos(180 \quad) \quad 98 \quad 10\sqrt{73}\cos$

Subtracting (2) from (1) and solving for cos ! we get cos ! = $\frac{\quad}{\sqrt{\quad}}$.

Substituting this value into equation (2) we get $PC^2 \quad 128$ from which PC = $8\sqrt{2}$.