THE 2014-2015 KENNESAW STATE UNIVERSITY
HIGH SCHOOL MATHEMATICS COMPETITION
PART II

Calculators are NOT

1. Let us say that a person is "acquainted" with another person if each of them knows the other. Suppose there is a gathering of 100 people and that among every four people at the gathering there is at least one who is acquainted with the other three. Prove that in this gathering, there is at least one person who is acquainted with all the remaining 99 people.

2. Suppose that and are integers such that + 2 and + 2 are squares. Prove that and are each multiples of 3.

3. Let P( ) be a polynomial with integer coefficients.

   (i) Prove that it is not possible for P(7) = 11 and P(11) = 13.

   (ii) Suppose that P(0) = P(1) = 1. Prove that the equation P( ) = 0 has no integer solutions.

4. In the rectangular coordinat [(In t)882 ([(In t)882 ([(In t)882 ([(In t)882 ([(In t)882 ([(I0 0.24 9In t)88

   . Let the four points of intersection of ℓ and ℓ' with and ′ be A, B, C, and D. Prove that points A, B, C, and D all lie on a circle.

5. Point P is chosen randomly on side AB of parallelogram ABCD. Line segment CP is extended to meet side DA extended at point M, and segment BM is constructed. Prove that the area of triangle BPM is equal to the area of triangle DAP.

1. Pick any person, say B. If B is acquainted with all the others, we are done.

   Otherwise, there is a person C such that B is not acquainted with C. Then C is not acquainted with

4. <u>Method 1</u> (elegant and simple!!)

We must show that quadrilateral ABCD is cyclic.

The lines intersect on the x and y-axes as shown, and the axes bisect the angles formed. Represent the origin as point T, and the intersections of and with the and -axes as S and R, respectively.

Let ! = m " APR = m " RPB and " = m " BQT = m " TQC.

Since " DAR is an exterior angle of #APR, we have m " DAR = ! + m " ARP.
Also, m " ARP = m " TRQ. Since #TRQ is a right triangle,
m " TRQ = 90 – m " RQT $ m " ARP = 90 – ". Therefore, m " DAR = ! + 90 – ".

The common solution to these two equations (the hypothetical center of the circle) is $P\left(-\dfrac{40}{63}, -\dfrac{68}{63}\right)$.

Now, rewriting the coordinates of points A, B, C, and D with a common denominator 63,

$$A\left(-\frac{56}{63}, \frac{140}{63}\right),\ B\left(\frac{72}{63}, \frac{108}{63}\right),\ C\left(\frac{168}{63}, -\frac{84}{63}\right), \text{ and } D\left(-\frac{216}{63}, -\frac{180}{63}\right),$$

and using the distance formula to find the lengths of $\overline{AP}$, $\overline{BP}$, $\overline{CP}$, and $\overline{DP}$, we obtain:

$$AP = \sqrt{\frac{16^2}{63^2} + \frac{(-208)^2}{63^2}},\ BP = \sqrt{\frac{112^2}{63^2} + \frac{176^2}{63^2}},\ CP = \sqrt{\frac{208^2}{63^2} + \frac{(-16)^2}{63^2}},\ DP = \sqrt{\frac{176^2}{63^2} + \frac{112^2}{63^2}}.$$

Since $16^2 + 208^2 = 43520 = 112^2 + 176^2$, A, B, C, and D are all the same distance from P.