**THE 2023-2024 KENNESAW STATE UNIVERSITY HIGH SCHOOL MATHEMATICS COMPETITIONPART II**

**Calculators are NOT permitted**

**Time allowed: 2 hours**

1. Let $(\ ) = 1 + {}_1 \quad + {}_2 {}^2 + \quad +$ where ${}_1, {}_2, \ldots,$ are integers and ${}_1 + {}_2 + \quad +$ is an even integer. Prove that there are no integer solutions to the equation $(\ ) = 0$.

2. Prove that in the set {2420, 2 2420, 3 2420}, there are exactly 44 numbers divisible by 2024.

3. Acute triangle ABC is inscribed in a circle. Altitudes $\overline{AM}$ and $\overline{CN}$ meet at point R and are extended to meet the circle at P and Q, respectively. If $PQ : AC = 8 : 5$, compute, with proof, the value of $\sin \quad ABC$.

   **Please include a diagram with your proof**.

4. Suppose we have two urns, each containing some red and some blue marbles, with at least one of each color in each urn. Assume that if we choose an urn randomly and then choose a marble randomly from that urn, then the probability of picking a red marble is the same as we would get by combining all the marbles into one urn and choosing a marble from that one at random. If the first urn contains 7 marbles and the second one contains 5 red marbles, find all possibilities for the number of marbles in the second urn. Prove that your answer is correct.

5. Prove that the hands of a clock (hour, minute, second hand) never trisect the face of the clock.

**Solutions**

1. Method 1:

   If $P(b) = 0$ for some integer $b$, then

   $1 + \;{}_{1}\; + \;{}_{2}\;{}^{2} + \;\; + \;\;\; = 0 \qquad (\;{}_{1} + \;{}_{2}\; + \;\; + \;\;\; {}^{-1}) = -1.$

   But this implies that $b$ divides 1 and the only integers that can do that are 1 and 1.

   If $b$ = 1, then $(\;) = 1 + \;{}_{1} + \;{}_{2} + \ldots +$ which is odd and so cannot be 0.

   If $b$ = 1, then $(\;) = 1 - \;{}_{1} + \;{}_{2} - \ldots + (-1)$

   $= (1 + \;{}_{1} + \;{}_{2} + \ldots + \;\;) - 2(\;{}_{1} + \;{}_{3} + \;{}_{5} + \;\;)$

   which is also odd and so cannot be 0. Therefore, there are no integer solutions to $P(x) = 0$.

   Method 2:

   If $P(b) = 0$ for some integer $b$: 9 Tf050.00000912 0 612 792 reW*nBT/F5 8.52 Tfp 7lTf1 0 0

   If b is even, then $1 + \;{}_{1}\; + \;{}_{2}\;{}^{2} + \;\; +$ , cannot be zero.

   If b is odd, $= 2\; + 1$ then $(2\; + 1) = 1 + 2 \;\; + \;{}_{1} + \;{}_{2} + \;\; +$ , cannot be zero.

2. Since (*2024*, *2420*) = *44*, then

   *2024* = *44s* and *2420* = *44r* for some integers *r* and *s* where (*r*, *s*) = 1 (i.e. *r* and *s* are relatively prime).

   Noting that $\frac{2420}{2024} = -$, if we divide each of the numbers in the given set by *2024*. We obtain the quotients:

   $\{\frac{2420}{2024}, \frac{2(2420)}{2024}, \ldots, \frac{2024(2420)}{2024}\} = -, \frac{2}{\;}, \ldots, \frac{(2024)}{\;} = -, \frac{2}{\;}, \ldots, ($ O

4. Denote the number of red marbles in the first urn by $r$ and the total number of marbles in the second by $t$.

Method 2:
The rate at which the angle between two hands of the clock changes is constant. As a result: if we see that at a time T hours later than 12:00, the minute hand is 120 degrees past the hour hand, then at a time 2T hours later than 12:00, the minute hand is 240 degrees past the hour hand, and at a time 3T hours later than 12:00, the minute hand is 360 degrees past the hour hand: they meet again.

Similarly, if at a time T hours later than 12:00, the second hand is 120 degrees past the minute hand, then at a time 3T hours later than 12:00, the second hand is 360 degrees past the minute hand - and they also meet. So, if we see Figure A (or, by a similar argument, Figure B) at time T, then at time 3T, we must see all three hands in the same place. This only happens every 12 hours.

Notice that the minute hand moves 12 times faster than the hour hand, so it passes the hour hand 11 times over the course of 12 hours: every time the hour hand makes 1/11 of a circle. Over that time, the second hand makes 65 full turns and another 5/11 of a circle, so it advances by 4/11 of a circle relative to the other two hands every time they meet. This has to repeat 11 times for the second hand to meet the other two, at which point it's 12:00 again.

Since 3T is a multiple of 12 hours, T must be a multiple of 4 hours, and we only need to investigate two times to see if they look like Figure A or Figure B: we need to check 4:00 and 8:00. However, we don't see Figure A or Figure B at those times: instead, the second hand and the minute hand meet at the top of the clock. Therefore, we never see Figure A or Figure B.